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NECESSARY WEIERSTRASS CONDITIONS FOR ELLIPTIC SYSTEMS†

L. V. PETUKHOV

St Petersburg

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Optimal control problems for objects described by systems of elliptic differential equations are considered. The controls are coefficients of leading terms in the equations and are constrained by equalities and inequalities. This formulation covers the optimal design problem for mechanical systems consisting of two materials with given volumes. The objective function is a functional which depends on the equations of state and their first derivatives. The main result is the construction of necessary Weierstrass conditions on inclusions of domains of one material in a domain of the other, which enables one to construct an approximation to the optimal solution. The case if a single n-dimensional elliptic equation is investigated in detail, for which a necessary Weierstrass condition is obtained when the inclusion is an n-dimensional ellipsoid. The application of a necessary Weierstrass condition to two optimal design problems is considered: the minimization of the work of external influences and the maximization of the torsional rigidity of a prismatic rod.

The traditional approach to the solution of optimal design problems consists of extending the set of admissible controls‡ within which the optimal solution exists. In most cases such a solution is of purely theoretical interest, and can be used to estimate the optimal value of the functional. Attempting to approximate it by using some kind of regular solution is impracticable because in the approximation one has to deal with a problem of high dimension.

This paper focuses on the problem of obtaining a necessary Weierstrass condition for a strong minimum, which, together with the necessary conditions for a weak minimum, can be used to obtain a progressive improvement in the connectedness of the domains of inclusion of one material in the other [1].

1. STATEMENT OF THE PROBLEM

Let $\mathbb{R}^n$ and $\mathbb{R}^m$ be n- and m-dimensional Cartesian vector spaces, respectively, i.e. real vector spaces of ordered sets of real numbers $x = (x_1, \ldots, x_n)$ and $u = (u_1, \ldots, u_m)$. Scalar products and norms in these spaces are defined in the usual manner

$$(x, y) = x_iy_i,\ \ |x| = (x, x)^{1/2},\ \ (u, \upsilon) = u_i\upsilon_i,\ \ |u| = (u, u)^{1/2}$$

Here and below the subscripts i and j take values from 1 to n, and the subscripts k, l take values from 1 to m. Repeated i or j indices in a product imply summation from 1 to n, and repeated k or l subscripts imply summation from 1 to m.

We denote by $\Omega \subset \mathbb{R}^n$ a regular domain with piecewise-smooth boundary Γ [2]. We assume that a vector function $f = (f_1(x), \ldots, f_m(x)) \in L^2(\Omega)$ is specified on Ω, together with a vector-function $F = (F_1(x), \ldots, F_m(x)) \in L^2(\Gamma_F)$ on $\Gamma_F \subset \Gamma$, where $L^2(\Omega)$, $L^2(\Gamma_F)$ are Hilbert spaces of vector functions with scalar products and norms

$$(f,g) = \int_\Omega f_k g_k dx,\ \ \|f\|_\Omega = \left(\int_\Omega |f|^2\, dx\right)^{1/2}$$

$$(F,G) = \int_{\Gamma_F} F_k G_k dx,\ \ \|F\|_{\Gamma_F} = \left(\int_{\Gamma_F} |F|^2\, d\Gamma\right)^{1/2}$$

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‡LUR'YE K. A. and CHERKEYEV A. V., Composition problem for an optimal isotropic multiphase composite. Preprint No. 895, Leningrad. Phys.-Techn. Inst., Academy of Sciences of the U.S.S.R., 1984.

In the domain Ω we consider vector-functions $u = (u_1(x), \ldots, u_m(x)) \in H^1(\Omega)$ where $H^1(\Omega)$ is the Sobolev space of vector functions with scalar product and norm

$$(u, \upsilon) = \int_\Omega (u_k \upsilon_k + u_{k,i} \upsilon_{k,i}) dx, \|u\| = (u, u)^{1/2}$$

containing the subspace

$$V(\Omega) = \{u \in H^1(\Omega) | u(x) = 0, \ x \in \Gamma_u\}$$

where $\Gamma_u \subset \Gamma$, $\Gamma_u \cap \Gamma_F = \varnothing$ and mes $\Gamma_u > 0$.

In $V(\Omega)$ we define the symmetric bilinear form $A(u, \upsilon) = a_{ijkl} u_{k,i} \upsilon_{ij}$ in which the coefficients a_{ijkl} possess the symmetry properties $a_{ijkl} = a_{jkil} = a_{iljk} = a_{jlik}$, and which satisfies the ellipticity condition

$$A(w, w) \geqslant \alpha \|w\|^2, \ \alpha > 0 \tag{1.1}$$

We define $u \in V(\Omega)$ to be the solution of the integral identity

$$\int_\Omega [A(u, w) - (f, w)] dx - \int_{\Gamma_F} (F, w) d\Gamma = 0, \ \forall w \in V(\Omega) \tag{1.2}$$

If $A(u, w)$ satisfies condition (1.1), and $f \in L^2(\Omega)$, $F \in L^2(\Gamma_F)$, then a unique solution $u \in V(\Omega)$ of the integral identity (1.2) exists [3].

We will now consider the formulation of the optimal design problem.

Suppose that the coefficients $a^{(1)}_{ikjl}$ and $a^{(2)}_{ikjl}$ are specified in measurable subspaces $\Omega_1 \subset \Omega$ and $\Omega_2 \subset \Omega$ which satisfy the conditions $\Omega_1 \cap \Omega_2 = \varnothing$, $\bar{\Omega}_1 \cup \bar{\Omega}_2 = \bar{\Omega}$ and

$$\text{mes } \Omega_1 = \lambda_1, \ \text{mes } \Omega_2 = \lambda_2 = \text{mes } \Omega - \lambda_1 \tag{1.3}$$

It is required to solve the problem

$$\inf J(u), \ J(u) = \int_{\Omega_1} \varphi(u, \sigma^{(1)}(u)) dx + \int_{\Omega_2} \varphi(u)) dx + \int_{\Gamma_F} \varphi(u) d\Gamma \tag{1.4}$$

where u satisfies the integral identity

$$\int_{\Omega_1} A_1(u, w) dx + \int_{\Omega_2} A_2(u, w) dx - \int_\Omega (f, w) dx - \int_{\Gamma_F} (F, w) d\Gamma = 0 \tag{1.5}$$
$$\forall w \in V(\Omega)$$

which is obtained from (1.2) and in which $A(u, w) = a^s_{ikjl} u_{k,i} w_{l,j}$. The functions φ and ψ in (1.4) are assumed to be differentiable n times with respect to all their arguments, and $\sigma^{(s)}(u)$ is the matrix with components $\sigma^{(s)}_{ik} = a^{(s)}_{ikjl} u_{l,j}$ $(s = 1, 2)$.

The sets Ω_1 and Ω_2 can have a fairly complicated structure and in certain cases can be found using averaging methods (cf. the publication cited in the footnote). The core of the analysis performed below is the determination of the sensitivity of functional (1.4) to inclusions of domains with coefficients $a^{(2)}_{ikjl}$ in the domain Ω_1 and domains with coefficients $a^{(1)}_{ikjl}$ in the domain Ω_2.

We shall now assume that in the domain Ω there are two regular domains Ω_1 and Ω_2 filled with the first and second materials respectively, that Γ_{12} is the boundary which separates the domains Ω_1 and Ω_2, and that $r = (r_1, \ldots, r_n)$ is the unit vector normal to it pointing from domain Ω_1 into domain Ω_2. We shall further assume that in the domains Ω_1 and Ω_2 the solution u to the integral identity (1.5) is a twice-differentiable vector-function with respect to the arguments x_i. Then, using the identity

$$a^{(s)}_{ikjl} u_{k,i} w_{l,j} = (a^{(s)}_{ikjl} u_{k,i} w_l)_{,j} - a^{(s)}_{ikjl} u_{k,ij} w_l$$

we obtain from (1.5) an elliptic boundary-value problem in the form of a system of differential equations with boundary conditions

$$a^{(s)}_{ikjl} u^{(s)}_{k,ij} + f_l = 0, \ x \in \Omega_s, \ s = 1, 2$$

$$u_k^{(s)}=0,\quad x\in\Gamma_u,\quad r_j\sigma_{jk}^{(s)}=F_k,\quad x\in\Gamma_F$$

$$r_j\sigma_{jk}^{(s)}=0,\quad x\in\Gamma_{s0}=\Gamma_s\setminus(\overline{\Gamma}_{12}\cup\overline{\Gamma}_F\cup\overline{\Gamma}_u)$$

$$u_k^{(1)}=u_k^{(2)},\quad r_j\sigma_{jk}^{(1)}=r_j\sigma_{jk}^{(2)},\quad x\in\Gamma_{12};\ \Gamma_{12}=\Gamma_1\cap\Gamma_2,\quad u=u^{(s)},\quad x\in\Omega_s$$

where Γ_1 and Γ_2 are respectively the boundaries of the domains Ω_1 and Ω_2.

2. NECESSARY OPTIMALITY CONDITIONS

We shall assume that there is an optimal solution such that the two domains Ω_1^* and Ω_2^* with common boundary $\Gamma_{12}^*(\Gamma_{12}^*=\Gamma_1^*\cap\Gamma_2^*)$ are filled with the first and second materials, respectively.

We construct an extended functional $I(u)$ in which the left-hand side of the identity (1.5) has been subtracted from the functional $J(u)$, and compute its first variation

$$\delta I=\chi_1(w,\delta u)+\chi_2(w,\delta u)+\int_{\Gamma_F}\frac{\partial\psi}{\partial u_k}\delta u_k d\Gamma+\int_{\Gamma_1^*\cap\Gamma}[\Lambda_1(u^*,w)-\Lambda_2(u^*,w)]\delta r d\Gamma+$$
$$+\int_{\Gamma_1^*\cap\Gamma}[\Lambda_2(u^*,w)-\Lambda_1(u^*,w)]\delta r d\Gamma+\int_{\Gamma_{12}^*}[\Lambda_1(u^*,w)-\Lambda_2(u^*,w)]\delta r d\Gamma \tag{2.1}$$
$$\chi_s(w,\delta u)=\int_{\Omega_s^*}\left[\frac{\partial\varphi}{\partial u_k}(u^*,\sigma^{(s)}(u^*))\delta u_k+\frac{\partial\varphi}{\partial\sigma_{ik}^{(s)}}(u^*,\sigma^{(s)}(u^*))\sigma_{ik}^{(s)}(\delta u)+A_s(w,\delta u)\right]dx,$$
$$\Lambda_s(u^*,w)=\varphi(u^*,\delta^{(s)}(u^*))-A_s(u^*,w),\quad s=1,2$$

Here u^* is the optimal solution of the problem, δu is the variation of u^*, and δr is a variation of the boundary Γ_{12}^* satisfying the condition [1]

$$\int_{\Gamma_{12}^*}\delta r d\Gamma+\int_{\Gamma_1^*\cap\Gamma}\delta r d\Gamma+\int_{\Gamma_2^*\cap\Gamma}\delta r d\Gamma=0 \tag{2.2}$$

which follows from condition (1.3).

In (2.1) we put $w=\upsilon^*$, where υ^* satisfies the integral identity

$$\chi_1(\upsilon^*,w)+\chi_2(\upsilon^*,w)+\int_{\Gamma_F}\frac{\partial\psi}{\partial u_k}(u^*)w_k d\Gamma=0,\quad\forall w\in V(\Omega) \tag{2.3}$$

Then, using $\delta u\in V(\Omega)$, we obtain the inequality

$$\delta I=\int_{\Gamma_1^*\cap\Gamma}[\Lambda_1(u^*,\upsilon^*]-\Lambda_2(u^*,\upsilon^*]\delta r d\Gamma+\int_{\Gamma_2^*\cap\Gamma}[\Lambda_1(u^*,\upsilon^*]-\Lambda_2(u^*,\upsilon^*]\delta r d\Gamma+$$
$$+\int_{\Gamma_{12}^*}[\Lambda_1(u^*,\upsilon^*]-\Lambda_2(u^*,\upsilon^*]\delta r d\Gamma\geqslant 0 \tag{2.4}$$

which must be satisfied by any variations of the boundary Γ_{12}^*, $\Gamma_1^*\cap\Gamma$, $\Gamma_2^*\cap\Gamma$ satisfying equality (2.2). It has been shown [2] that the conditions

$$\Lambda_1(u^*,\upsilon^*)-\Lambda_2(u^*,\upsilon^*)\leqslant\zeta,\quad x\in\Gamma_1^*\cap\Gamma$$
$$\Lambda_1(u^*,\upsilon^*)-\Lambda_2(u^*,\upsilon^*)\geqslant\zeta,\quad x\in\Gamma_2^*\cap\Gamma$$
$$\Lambda_1(u^*,\upsilon^*)-\Lambda_2(u^*,\upsilon^*)=\zeta,\quad x\in\Gamma_{12}^* \tag{2.5}$$

are necessary and sufficient for (2.2) and (2.4) to be satisfied.

3. NECESSARY WEIERSTRASS CONDITIONS

We will analyse a variation of the functional in which a small domain of the first material is included in the domain of the second material, and vice versa.

We take $\forall x_0 \in \Omega_1^*(\forall x_0 \in \Omega_2^*)$ and a convex domain $\Omega_0(\eta)$ all of whose geometrical dimensions can vary in proportion to η. Then

$$\text{mes}\ \Omega_0(\eta) = \eta^n\ \text{mes}\ \Omega_0(\Omega_0 = \Omega_0(1)) \tag{3.1}$$

In order for condition (1.3) to remain satisfied, it is necessary to vary the boundary Γ_{12}^* by an amount $r(y, \eta), y \in \Gamma_{12}^*$ [2], where $r(y, \eta) < 0$ in the case of an inclusion of the first material in the second, and $r(y, \eta) > 0$ in the case of an inclusion of the second material in the first.

In this case functional (1.4) can be reduced to the form

$$I = \int_{\Gamma_F} [\psi(u) - (F, w)]d\Gamma - \int_{\Omega} (f, w)dx + \int_{\Omega_1} \Lambda_1(u, w)dx +$$
$$+ \int_{\Omega_2} \Lambda_2(u, w)dx \pm \int_{\Omega_0(\eta)} [\Lambda_1(u, w) - \Lambda_2(u, w)]dx \tag{3.2}$$

The plus sign in front of the last integral corresponds to the inclusion of the first material in the second, and the minus sign to the inclusion of the second material in the first.

In order to satisfy condition (1.3) the function $r(y, \eta)$ must be proportional to η^n, hence $\delta r = .. \delta^{n-1} r = 0$. The variations $\delta u, \ldots, \delta^n u$ are given by integral identities in which the external actions are functions proportional to $\delta r, \ldots, \delta^n r$, and hence $\delta u = \ldots = \delta^{n-1} u = 0$ [1].

Substituting $w = \upsilon^*$ into functional (3.2) and sequentially computing the variations $\delta I, \ldots, \delta^n I$, we obtain

$$\delta I = \ldots = \delta^{n-1} I = 0$$

$$\delta^n I = \int_{\Gamma_{12}^*} [\Lambda_1(u^*, \upsilon^*) - \Lambda_2(u^*, \upsilon^*)]\delta^n r d\Gamma \pm \frac{d^n}{d\eta^n}\left(\int_{\Omega_0(\eta)} [\Lambda_1(u^*, \upsilon^*) - \Lambda_2(u^*, \upsilon^*)]dx\right)_{\eta=0}$$

Using the necessary conditions $\delta^n I \geqslant 0$ and (2.5) we find

$$\pm \frac{d^n}{d\eta^n} \int_{\Omega_0(\eta)} [\Lambda_1(u^*, \upsilon^*) - \Lambda_2(u^*, \upsilon^*)]dx \geqslant \pm n! \zeta \text{mes} \Omega_0 \tag{3.3}$$

The minus sign is taken for points $x_0 \in \Omega_1^*$ and the plus sign for $x_0 \in \Omega_2^*$.

In the left-hand side of inequality (3.3) the integral is evaluated over the domain $\Omega_0(\eta)$ (an n-dimensional volume proportional to η^n) of an expression which depends on the vector function u, which is a perturbation of the solution to the integral identity (1.5) that has been produced by the inclusion $\Omega_0(\eta)$.

We will obtain integral identities for determining u. To do this we consider an inclusion $\Omega_0(\eta)$ of the second material in the domain Ω_1^*. Subtracting from the integral identity for this case the integral identity for the optimal solution u^* and transforming it, we find

$$\int_{\Omega_1^* \setminus \bar{\Omega}_0(\eta)} A_1(u^0 - u^*, w)dx + \int_{\Omega_0} (u^0 - u^*, w)dx = \int_{\partial\Omega_0} r_i[\sigma_{il}^{(1)}(u^*(x_0)) - \sigma_{il}^{(2)}(u^*(x_0))]w_l d\Gamma,$$
$$\forall x_0 \in \Omega_1^* \tag{3.4}$$

It is of course impossible to determine u exactly in the integral identity (3.4). However, when $\eta \to 0$ we have $u \to u^0$ for which the integral identity

$$\int_{\mathbb{R}^n \setminus \Omega_0} A_1(u^0 - u^*, \omega)dx + \int_{\Omega_0} A_2(u^0 - u^*, \omega)dx = \int_{\partial\Omega_0} r_i[\sigma_{il}^{(1)}(u^*(x_0)) - \sigma_{il}^{(2)}(u^*(x_0))]\omega_l d\Gamma,$$
$$\forall x_0 \in \Omega_1^* \tag{3.5}$$

holds, where the r_i are the components of the external unit normal to the boundary $\partial\Omega_0$ of the domain Ω_0.

Performing similar constructions for an inclusion $\Omega_0(\eta)$ of the first material into the second, we obtain an integral identity differing from (3.5) by the replacement of the superscript 1 by the superscript 2 and vice versa.

Using the latter two identities together with inequality (3.3), we finally obtain the inequalities

$$\begin{aligned}
&\Lambda_1(u^0,\upsilon^0)-\Lambda_2(u^0,\upsilon^*)\leqslant\zeta,\ \ x\in\Omega_1^*\\
&\Lambda_1(u^0,\upsilon^0)-\Lambda_2(u^0,\upsilon^*)\geqslant\zeta,\ \ x\in\Omega_2^*\\
&u^*=\upsilon^*(x_0),\ \ \upsilon^*=\upsilon^*(x_0),\ \ u^0=u^0(x_0)
\end{aligned}\tag{3.6}$$

Inequalities (3.6) are called necessary Weierstrass conditions. The first must be satisfied by any $x_0 \in \Omega_1^*$ and the second for any $x_0 \in \Omega_2^*$.

4. NECESSARY WEIERSTRASS CONDITIONS FOR AN ELLIPSOIDAL INCLUSION ($m = 1$)

For the case when $m = 1$ the bilinear form $A(u, \upsilon) = a_{ij}u_{,i}\upsilon_{,j}$ can be reduced to the form

$$A(u,\upsilon)=hu_{,i}\upsilon_{,i}\tag{4.1}$$

by a change of coordinates.

Below we shall assume that the Cartesian system of coordinates x_i has been chosen from the start so that the bilinear form $A(u, \upsilon)$ has the form (4.1), and also $h_2 > h_1$. The components of the vectors $\sigma^{(s)}$ are given by the relations $\sigma_i^{(1)}(u) = h_1u_{,i}$, $\sigma_i^{(2)}(u) = h_2u_{,i}$ and the necessary Weierstrass conditions (3.6) are the inequalities

$$\begin{aligned}
&\Delta h\,u^0_{,i}\upsilon^*_{,i}-\varphi(u^*,\sigma^{(2)}(u^0))+\varphi(u^*,\sigma^{(1)}(u^0))\leqslant\zeta\\
&\Delta h\,u^0_{,i}\upsilon^*_{,i}-\varphi(u^*,\sigma^{(2)}(u^0))+\varphi(u^*,\sigma^{(1)}(u^0))\geqslant\zeta
\end{aligned}\tag{4.2}$$

where $\Delta h = h_2 - h_1$.

In this case the solution u^0 of the integral identities (3.5) and (3.6) are known [4]

$$\begin{aligned}
&u^0=u^*+\kappa_i^{(s)}x_iu^*_{,i}(x_0),\ \ s=1,2\\
&\kappa_i^{(1)}=-\Delta h\mu_i(h_1+\Delta h\mu_i)^{-1},\ \ \kappa_i^{(2)}=\Delta h\mu_i(h_2+\Delta h\mu_i)^{-1}\\
&\mu_i=\frac{1}{2}a_1\ldots a_n\int_0^\infty\frac{d\rho}{(a_i^2+\rho)\sqrt{(a_1^2+\rho)\ldots(a_n^2+\rho)}}
\end{aligned}\tag{4.3}$$

where the superscript 1 in parentheses corresponds to an inclusion of the second material in the first and the 2 corresponds to the first material in the second.

Without loss of generality we can assume that the semi-axes of the ellipsoid have been ordered as follows:

$$a_1=1,\ \ a_1\geqslant a_2\geqslant\ldots a_n\geqslant0\tag{4.4}$$

It then follows from (4.3) that $0 \leqslant \mu_1 \leqslant \mu_2 \leqslant \ldots \mu_n \leqslant 1$, the last inequality being obtained from an estimate of the integrand in integral (4.3)

$$\mu_n\leqslant\frac{a_n}{2}\int_0^\infty\frac{d\rho}{(a_n^2+\rho)^{3/2}}=1$$

Analysis shows that $\kappa^{(1)}(\mu)$ is a monotonically decreasing, and $\kappa^{(2)}(\mu)$ is a monotonically increasing function of μ when $0 \leqslant \mu \leqslant 1$ (Fig. 1).

We will consider the case when function $\varphi = \varphi(u)$. Then inequalities (4.2) take the respective forms

$$\begin{aligned}
&\Delta hu^*_{,i}\upsilon^*_{,i}\leqslant\zeta\pm\Delta h\kappa_i^{(1)}u^*_{,i}\upsilon^*_{,i},\ \ \forall x_0\in\Omega_1^*\\
&\Delta hu^*_{,i}\upsilon^*_{,i}\geqslant\zeta\pm\Delta\kappa_i^{(2)}u^*_{,i}\upsilon^*_{,i},\ \ \forall x_0\in\Omega_2^*
\end{aligned}\tag{4.5}$$

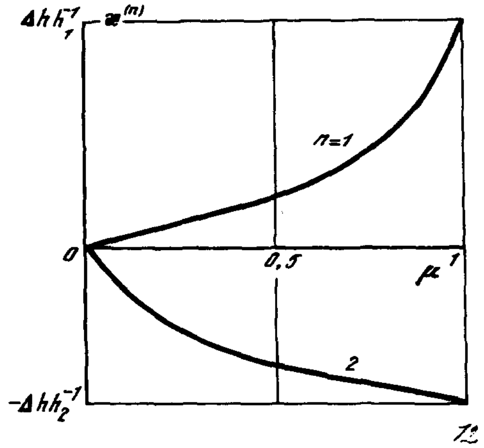

Fig. 1.

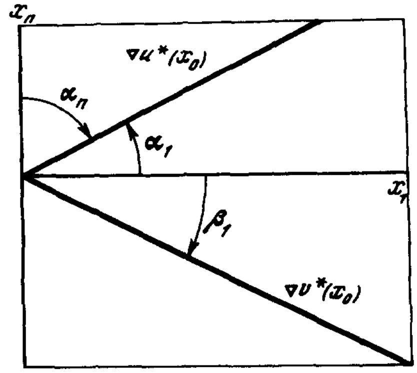

Fig. 2.

(where $u^* = u^*(x_0)$, $\upsilon^* = \upsilon^*(x_0)$). We denote by α_i the angles between $\nabla u^*(x_0)$—the gradient of the function u^* at the point x_0—and the axes of the n-dimensional ellipsoid, by β_i the angles between $\nabla u^*(x_0)$ —the gradient of the function υ^* at the point x_0—and the axes of the n-dimensional ellipsoid, and by γ the angle between the gradients $\nabla u^*(x_0)$ and $\nabla u^*(x_0)$. Then inequalities (4.5) can be represented in the form

$$\begin{aligned} &\Delta h|\nabla u^*||\nabla \upsilon^*|\cos\gamma \leqslant \zeta - \Phi_1(\alpha_i, \beta_i, a_i) \\ &\Delta h|\nabla u^*||\nabla \upsilon^*|\cos\gamma \leqslant \zeta - \Phi_2(\alpha_i, \beta_i, a_i) \\ &\Phi_s = \Delta h|\nabla u^*||\nabla \upsilon^*|\kappa_i^{(s)}\cos\alpha_i\cos\beta_i, \quad s = 1,2 \end{aligned} \tag{4.6}$$

Inequalities (4.6) must be satisfied by all α_i, β_i, a_i satisfying the equalities

$$\sum_{i=1}^{n}\cos^2\alpha_i = 1, \quad \sum_{i=1}^{n}\cos^2\beta_i = 1, \quad \sum_{i=1}^{n}\cos\alpha_i\cos\beta_i = \cos\gamma \tag{4.7}$$

and inequalities (4.4).

We will solve the two problems

$$\max_{\alpha_i,\beta_i,a_i}\Phi_1(\alpha_i,\beta_i,a_i), \quad \min_{\alpha_i,\beta_i,a_i}\Phi_2(\alpha_i,\beta_i,a_i)$$

for α_i, β_i, a_i satisfying (4.7) and (4.4). Then inequalities (4.6) hold for all α_i, β_i, a_i satisfying conditions (4.7) and (4.4). The maximum of the function $\Phi_1(\alpha_i, \beta_i, a_i)$ and the minimum of the function $\Phi_2(\alpha_i, \beta_i, a_i)$ are reached at the points (Fig. 2)

$$\begin{aligned} &a_1 = \ldots = a_{n-1} = 1, \ a_n = 0, \ \alpha_1 = -\beta_1 = \gamma/2 \\ &\alpha_2 = \beta_2 = \ldots = \alpha_{n-1} = \beta_{n-1} = \pi/2, \ \alpha_n = \pi/2 - \gamma/2, \ \beta_n = \pi/2 + \gamma/2 \end{aligned} \tag{4.8}$$

and are given by

$$\Phi_1^* = h_2^{-1}\Delta h^2|\nabla u^*||\nabla \upsilon^*|\sin^2(\gamma/2), \quad \Phi_2^* = -h_1^{-1}\Delta h^2|\nabla u^*||\nabla \upsilon^*|\sin^2(\gamma/2)$$

Analysis of relations (4.8) shows that the extremal values of the functions Φ_1 and Φ_2 are reached on an n-dimensional oblate spheroid the normal to which coincides with the x_n axis. The necessary Weierstrass conditions finally take the form

$$\Delta h|\nabla u^*||\nabla \upsilon^*|\cos\gamma \leqslant \zeta - h_2^{-1}\Delta h^2|\nabla u^*||\nabla \upsilon^*|\sin^2(\gamma/2) \tag{4.9}$$
$$\Delta h|\nabla u^*||\nabla \upsilon^*|\cos\gamma \geqslant \zeta + h_1^{-1}\Delta h^2|\nabla u^*||\nabla \upsilon^*|\sin^2(\gamma/2)$$

where $u^* = u^*(x_0)$, $\upsilon^* = \upsilon^*(x_0)$.

Example 1. Consider the problem of minimizing the work of f and F through a displacement u. In this case $\varphi = fu$, $\psi = Fu$. Analysis of integral identities (1.5) and (2.3) shows that $\gamma = 0$, and the necessary Weierstrass conditions consequently have the simple form

$$\Delta h|\nabla u^*|^2 \leqslant \zeta,\ x_0 \in \Omega_1^*;\ \Delta h|\nabla u^*|^2 \geqslant \zeta,\ x_0 \in \Omega_2^* \tag{4.10}$$

We put $x_0 = y_- = y - 0r$ in the first inequality and $x_0 = y_+ = y + 0r$ in the second, where r is the unit normal to the boundary Γ_{12}^*, and $y \in \Gamma_{12}^*$. We also note that $\zeta = h_2 |\nabla u^*|^2 (y_+) |^2 - h_1 |\nabla u^* (y_-)|^2$. Analysis of inequalities (4.10) shows that they can be satisfied if

$$\partial u^*(y_-)/\partial r = \partial u^*(y_+)/\partial r = 0,\ \forall y \in \Gamma_{12}$$

Example 2. Consider the problem of maximizing the torsional rigidity of a rod composed of two materials, one flexible and one stiff, with shear moduli G_1 and G_2, respectively. In this case $\varphi = -2u$, $f = 2$, $h_1 = \rho G_1$, $h_2 = \rho G_2$, where the quantity ρ is proportional to the torque applied to the rod. It follows from integral identities (1.5) and (2.3) that $\upsilon^* = u^*$. In this case $\gamma = \pi$, $\zeta = -\Delta |\nabla u^*(y)|^2$ for $y \in \Gamma_{12}^*$, and inequalities (4.9) acquire the form

$$\begin{aligned} &h_1 h_2^{-1}|\nabla u^*(x_0)|^2 \geqslant \zeta,\ \forall x_0 \in \Omega_1^* \\ &h_2 h_1^{-1}|\nabla u^*(x_0)|^2 \leqslant \zeta,\ \forall x_0 \in \Omega_2^* \end{aligned} \tag{4.11}$$

We put $x_0 = y_- = y - 0r$ in the first inequality and $x_0 = y_+ = y + 0r$ in the second, where $y \in \Gamma_{12}^*$ and r is the unit normal to Γ_{12}^*. Then from (4.11) one can obtain the inequality

$$h_1(2h_2 - h_1)|\nabla u^*(y_-)|^2 \geqslant h_2^2|\nabla u^*(y_+)|^2$$
$$h_2(2h_1 - h_2)|\nabla u^*(y_+)|^2 \geqslant h_1^2|\nabla u^*(y_-)|^2$$

from which it follows that they can only be satisfied in the case when

$$u_{,i}^*(y_-) = u_{,i}^*(y_+) = 0,\ i = 1,\ldots,n$$

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